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Universal *R* Matrix of Two-Parameter Deformed Quantum Group $U_{qs}(SU(1, 1))$

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The universal R matrix of the two-parameter deformed quantum group $U_{qs}(SU(1, 1))$ is derived. In previous work we suggested a method to derive the universal R matrix of the two-parameter deformed quantum group $U_{qs}(SU(2))$. This method is different from that of the quantum double; it is simple and efficient for quantum groups of low rank at least. This paper studies the universal R matrix of the two-parameter deformed quantum group $U_{qs}(SU(1, 1))$ using the same approach.

The two-parameter deformed quantum group $U_{qs}(SU(1, 1))$ has three unitary irreducible representations (Jing and Cuypers, 1993): a positive discrete series, a negative series, and a continuous series. The generators of the two-parameter deformed quantum group $U_{qs}(SU(1, 1))$ can be obtained from a Jordan–Schwinger realization in terms of two-parameter deformed bosonic creation and annihilation operators:

$$L_+^a = s^{-1} a_1^+ a_2^+, \qquad L_-^a = s^{-1} a_1 a_2, \qquad L_0^a = \tfrac{1}{2}(N_1^a + N_2^a + 1) \tag{1}$$

$$L_+^b = s b_1 b_2, \qquad L_-^b = s b_1^+ b_2^+, \qquad L_o^b = \frac{-1}{2}(N_1^b + N_2^b + 1) \tag{2}$$

where $\{a_i^+, a_i, N_i^a\}$ and $\{b_i^+, b_i, N_i^b\}$ $(i = 1, 2)$ are independent and satisfy the commutation relations

$$a_i^+ a_i = [N_i^a]_{qs}, \qquad a_i a_i^+ = [N_i^a + 1]_{qs} \tag{3}$$

$$b_i^+ b_i = [N_i^b]_{qs^{-1}}, \qquad b_i b_i^+ = [N_i^b + 1]_{qs^{-1}} \tag{4}$$

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where the deformation brackets are defined as

$$[x]_{qs} = s^{1-x}[x] = s^{1-x}(q^x - q^{-x})/(q - q^{-1}), \qquad [x]_{qs^{-1}} = s^{x-1}[x] \quad (5)$$

It is easy to check that

$$[L_o^{a(b)}, L_\pm^{a(b)}] = \pm L_\pm^{a(b)}, \qquad s^{-1}L_+^{a(b)} L_-^{a(b)} - sL_-^{a(b)} L_+^{a(b)} = -s^{-2L_o^{a(b)}}[2L_0^{a(b)}] \quad (6)$$

For simplicity, we will omit the index $a(b)$ in the following discussion. The quantum $U_{qs}(SU(1, 1))$ is a Hopf algebra; its coproduct is (Yu *et al.*, 1996, 1997a, b)

$$\Delta(L_0) = L_0 \otimes 1 + 1 \otimes L_0 \quad (7)$$

$$\Delta(L_\pm) = L_\pm \otimes (sq)^{-L_0} + (s^{-1}q)^{L_0} \otimes L_\pm \quad (8)$$

We define an inverse of the coproduct $\overline{\Delta} = T\,\Delta$, where T is the twisted mapping, i.e.,

$$T(x \otimes y) = y \otimes x, \qquad \forall x, y \in U_q(SU(1, 1)) \quad (9)$$

So the following relation holds:

$$\overline{\Delta}(a)R = R\Delta(a), \qquad a \in U_q(SU(1, 1)) \quad (10)$$

with R is the universal matrix and can be written as

$$R = \sum_i a_i \otimes b_i \quad (11)$$

Accordingly, Eq. (11) satisfies the Yang–Baxter equation (Yang, 1967; Baxter, 1972)

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \quad (12)$$

where

$$R_{12} = R \otimes 1, \qquad R_{13} = \sum_i a_i \otimes 1 \otimes b_i, \qquad R_{23} = 1 \otimes R$$

For convenience, let x and x' stand for the first and the second operator in the tensor product $U_{qs}(SU(1, 1)) \otimes U_{qs}(SU(1, 1))$, respectively. Therefore Eqs. (7), (8), and (10) take the form, respectively.

$$\Delta(L_0) = L_0 + L_0' \quad (13)$$

$$\Delta(L_\pm) = L_\pm(sq)^{-L_0'} + (s^{-1}q)^{L_0} L_\pm' \quad (14)$$

$$\overline{\Delta}(L_0)R(x, x') = R(x, x')\Delta(L_0) \quad (15)$$

$$\overline{\Delta}(L_\pm)R(x, x') = R(x, x')\Delta(L_\pm) \quad (16)$$

In order to get the solution of Eqs. (15) and (16), we let

$$R(x, x') = \sum_{l=0}^{\infty} C_l(L_0, L_0')L_-^l L_+'^l \tag{17}$$

where $C_l(L_0, L_0')$ is a functional of the operators L_0 and L_0' as well as paramart-ers l, q, and s.

To obtain nontrivial results, we have to substitute Eq. (17) into Eq. (16):

$$s^{-L_0'+l}q^{L_0'+l}C_l(L_0 - l, L_0' + l) = (sq)^{-L_0'}C_l(L_0 - l + 1, L_0' + l) \tag{18}$$

$$(sq)^{-L_0+l}C_l(L_0 - l, L_0' + l) - (s^{-1}q)^{L_0}C_l(L_0 - l, L_0' + l + 1)$$
$$= s^{-2L_0-L_0'}q^{L_0'+l+1}[l+1]_{qs^{-1}}[2L_0 - l]C_{l+1}(L_0 - l - 1, L_0' + l + 1) \tag{19}$$

$$s^{-L_0-l}q^{-L_0+l}C_l(L_0 - l, L_0' + l) = (s^{-1}q)^{L_0}C_l(L_0 - l, L_0' + l - 1) \tag{20}$$

$$(sq)^{-L_0'}C_l(L_0 - l - 1, L_0' + l) - (s^{-1}q)^{L_0'+l}C_l(L_0 - l, L_0' + l)$$
$$= s^{-L_0-2L_0'}q^{-L_0+l+1}[l+1]_{qs}[2L_0' + l]\, C_{l+1}(L_0 - l - 1, L_0' + l + 1) \tag{21}$$

We let

$$C_l(L_0, L_0') = \tilde{C}_l s^{aL_0L_0'+bL_0+cL_0'}q^{dL_0L_0'+eL_0+fL_0'} \tag{22}$$

On the substitution of Eq. (22) into Eqs. (18) and (20, respectively, we have

$$a = 0, \qquad b = c = l, \qquad d = 2, \qquad e = -l, \qquad f = l \tag{23}$$

The recurrence formula is easy to get

$$\tilde{C}_l = (q^{-2} - 1)^l q^{-l(l-1)/2}/[l]! \tag{24}$$

where we have $\tilde{C}_0 = 1$. Equation (17) can be rewritten as

$$R(x, x') = \sum_{l=0}^{\infty} \frac{(q^{-2} - 1)^l q^{-l(l-1)/2}}{[l]!} s^{l(L_0+L_0')} q^{2L_0L_0'-l(L_0-L_0')} L_-^l L_+'^l \tag{25}$$

Let us check whether Eq. (25) holds for the Yang–Baxter equation,

$$R(x, x')R(x, x'')R(x', x'') = R(x', x'')R(x, x'')R(x, x') \tag{26}$$

The left-hand side of Eq. (26) is

$$R(x, x')R(x, x'')R(x', x'')$$
$$= \sum_{M,N=0}^{\infty} q^{2L_0L_0' + 2L_0L_0'' + 2L_0'L_0''} q^{-M(L_0-L_0')-N(L_0'-L_0'')+NM}$$

$$\times\, s^{M(L_0+L_0')+N(L_0'+L_0'')} L_-^M L_+^{''N} \sum_{l=0}^{\min(M,N)} \tilde{C}_{M-l}\tilde{C}_l\tilde{C}_{N-l} q^{-l(2L_0'+2N-l)}$$

$$\times\, s^{-l(2L_0'-2M+l)-NM} L_+^{'M-l} L_+^{'N-l} \tag{27}$$

The right-hand side of Eq. (26) is

$R(x', x'')R(x, x'')R(x, x')$

$$\sum_{M,N=0}^{\infty} q^{2L_0L_0'+2L_0L_0''+2L_0'L_0''} q^{-M(L_0-L_0')-N(L_0'-L_0'')+NM}$$

$$\times\, s^{M(L_0+L_0')+N(L_0'+L_0'')} L_-^M L_+^{''N} \sum_{L=0}^{\min(M,N)} \tilde{C}_{N-l}\tilde{C}_l\tilde{C}_{M-l} q^{l(2L_0'-2M+l)}$$

$$\times\, s^{l(-2L_0'-2N+l)+NM} L_-^{'N-l} L_+^{'M-l} \tag{28}$$

On the other hand, we have for all nonnegative integers M and N

$$\sum_{l=0}^{\min(M,N)} \tilde{C}_{M-l}\tilde{C}_l\tilde{C}_{N-l} q^{-l(2L_0'+2N-l)} s^{-l(2L_0'-2M+l)-NM} L_+^{'M-l} L_-^{'N-l}$$

$$= \sum_{l=0}^{\min(M,N)} \tilde{C}_{N-l}\tilde{C}_l\tilde{C}_{M-l} q^{l(2L_0'-2M+l)} s^{l(-2L_0'-2N+l)+NM} L_-^{'N-l} K_+^{'M-l} \tag{29}$$

From Eqs. (27)–(29), we conclude that Eq. (26) is the universal R matrix of the two-parameter deformed quantum group $U_{qs}(SU(1,\ 1))$.

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